ON THE THEORY OF UNSTEADY SUPERSONIC GAS FLOW ABOUT A WING OF FINITE SPAN

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PMM Vol.24, No.1, 1960, pp. 166-168

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(Received 8 April 1959)

1. As is well-known, the unsteady three-dimensional potential motion of an inviscid compressible fluid is described in vector form by the following equation:

$$\frac{\partial^2 \varphi}{\partial t^2} - a^2 \operatorname{div} \mathbf{c} + \mathbf{c} \operatorname{grad}\left(2\frac{\partial \varphi}{\partial t} + \frac{\mathbf{c}^2}{2}\right) = 0 \qquad (1.1)$$

where a is the velocity of sound and $\mathbf{c}$ is the velocity vector of the air flow.

Assuming that the air flows about the wing with constant velocity c in the positive direction along the x-axis and that the disturbances introduced by the wing are small, the perturbation velocity potential ϕ_1 will satisfy the wave equation in a rectangular coordinate system

$$\beta^2 \frac{\partial^2 \varphi_1}{\partial x^2} - \frac{\partial^2 \varphi_1}{\partial y^2} - \frac{\partial^2 \varphi_1}{\partial z^2} + \frac{1}{a^2}\frac{\partial^2 \varphi_1}{\partial t^2} + 2\frac{M}{a}\frac{\partial^2 \varphi_1}{\partial x \partial t} = 0 \qquad (\beta = \sqrt{M^2 - 1}) \qquad (1.2)$$

Here M is the Mach number of the undisturbed flow.

As a consequence of its linearity the solution of this equation is the expression

$$\varphi_1(x, y, z, t) = -\frac{1}{2\pi}\iint\limits_S \frac{1}{r}\left[q(\xi, \eta, t - \tau_1) + q(\xi, \eta, t - \tau_2)\right] d\xi d\eta \qquad (1.3)$$

where $q(x, y, t)$ is the source distribution on the surface of the wing,

$$r = \sqrt{(x-\xi)^2 - \beta^2[(y-\eta)^2 + z^2]}, \qquad \tau_{1,2} = \frac{M}{a\beta^2}(x - \xi) \mp \frac{r}{a\beta^2} \qquad (1.4)$$

The region of integration S lies within the forward characteristic Mach cone whose apex is at the given point.

The value of the function for a displacement of its argument by an amount τ can be determined with the help of the following exponential differential operator:

$$f(t \pm \tau) = \exp\left(\pm \tau \frac{\partial}{\partial t}\right) f(t)$$

Using this operator on the source distribution and keeping in mind the notation of (1.4), the expression for the velocity potential ϕ_1 can be represented as

$$\varphi_1 = -\frac{1}{\pi}\iint\limits_S \frac{1}{r} \exp\left(\mu M (x-\xi) \frac{\partial}{\partial t}\right) \operatorname{ch}\left(\mu r \frac{\partial}{\partial t}\right) q(\xi, \eta, t)\, d\xi d\eta \qquad \left(\mu = \frac{1}{a\beta^2}\right) \tag{1.5}$$

After writing the hyperbolic cosine and the exponential function as the series expansions

$$\operatorname{ch}\left(\mu r \frac{\partial}{\partial t}\right) = \sum_{m=0}^{\infty} \frac{(\mu r)^{2m}}{(2m)!} \frac{\partial^{2m}}{\partial t^{2m}} \qquad \exp\left(\mu M (x-\xi) \frac{\partial}{\partial t}\right) = \sum_{m=0}^{\infty} \frac{[\mu M (x-\xi)]^m}{m!} \frac{\partial^m}{\partial t^m}$$

formula (1.5) can be represented as

$$\varphi_1(x, y, z, t) = -\frac{1}{\pi}\iint\limits_S \sum_{m=0}^{\infty} \frac{[\mu M (x-\xi)]^m}{m!} \frac{\partial^m}{\partial t^m} \sum_{m=0}^{\infty} \frac{\mu^{2m}}{(2m)!} r^{2m-1} \frac{\partial^{2m}}{\partial t^{2m}} q(\xi, \eta, t)\, d\xi d\eta \tag{1.6}$$

We will represent the double series under the integral sign in the form of a series with respect to the orders of the derivatives. Then we finally obtain

$$\varphi_1(x, y, z, t) = -\frac{1}{\pi}\sum_{m=0}^{\infty} \frac{\partial^m}{\partial t^m} \sum_{k=n}^{k=m} \frac{\mu^m M^{2k-m}}{[2(m-k)]!\,(2k-m)!} \iint\limits_S (x-\xi)^{2k-m} r^{2(m-k)-1} q(\xi, \eta, t)\, d\xi d\eta \tag{1.7}$$

where $n = 1/2\, m$ if m is an even number, and $n = 1/2(m+1)$ if m is an odd number.

The pressure difference on the top and bottom of the wing at any point is equal to

$$\Delta p(x, y, t) = -2\rho\left(\frac{\partial\varphi}{\partial t} + c\frac{\partial\varphi}{\partial x}\right) \tag{1.8}$$

The value $m = 0$ corresponds to the steady theory and the velocity potential in this case (remembering that $0! = 1$) is

$$\varphi_1(x, y, z, t) = -\frac{1}{\pi}\iint\limits_S \frac{1}{r} q(\xi, \eta, t)\, d\xi d\eta \tag{1.9}$$

Retention of the first two terms of the series corresponds to the quasi-steady theory. In the case of periodic motion the operator $\partial/\partial t = i\omega$

(where ω is the circular frequency of the oscillation). This case is examined in detail in [1].

2. The result obtained can be applied to the solution of problems associated with the determination of the aerodynamic forces on an elastic wing of finite span resulting from the action of an arbitrary aperiodic disturbance. In particular, we will apply the given method to the calculation of the air loads which act on an elastic aircraft as it encounters a vertical air gust of arbitrary spatial form. For this the wing is represented as a free, elastic plate symmetric with respect to the x-axis and having the origin of the coordinate system located in its leading edge. Elastic deformations in a coordinate system fixed in the aircraft are assumed in the form

$$z = \sum_{j=1}^{\infty} w_j(x, y)\, p_j(t) \tag{2.1}$$

where $w_j(x, y)$ is the form of the oxcillations of the jth mode and $p_j(t)$ are unknown functions of the time.

Then the source distribution in a new fixed-in-space system of coordinates is expressed in the following form:

$$q(x, y, t) = \frac{dz_0}{dt} - (x - x_0)\frac{d\theta}{dt} + c\theta + z\frac{d\psi}{dt} + \sum_{j=1}\left(w_j\frac{dp_j}{dt} - cp_j\frac{\partial w_j}{\partial x}\right) + W(x, y, t) \tag{2.2}$$

where z_0 is the vertical deformation of the center of gravity of the plate, x_0 is the coordinate of the center of gravity, $\theta(t)$ is the angle of pitch, $\psi(t)$ is the angle of roll, and $W(x, y, t)$ is the velocity of the vertical gust.

For practical problems it is sufficient in the majority of cases to restrict oneself to three or four forms of the elastic oscillations ($j = 1, 2, 3, 4$). Further, the pressure difference on the wing is determined from formula (1.8) and the system of differential equations for the motion of the aircraft is formed. Limiting oneself to a certain number of terms of the series (1.7) depending on the capabilities of the electronic computer, the unknown functions $p_j(t)$, $\theta(t)$, $z_0(t)$ and $\psi(t)$ can be determined.

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Translated by R.D.C.